

ECE750T-28: Computer-aided Reasoning for Software Engineering

Lecture 16: Decision Procedures for Combination Theories

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(Original notes from Isil Dillig)

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- ▶ This formula does not belong to any individual theory
- ▶ But it does belong, for instance, to combination of $T_{=}$ and $T_{\mathbb{Z}}$

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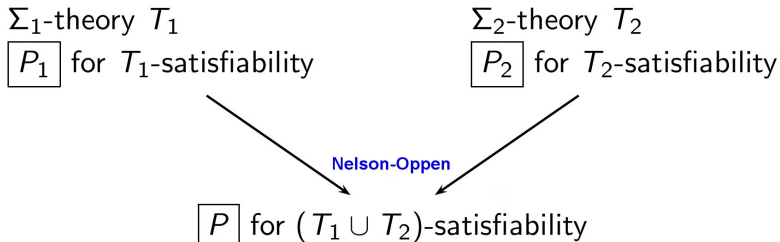
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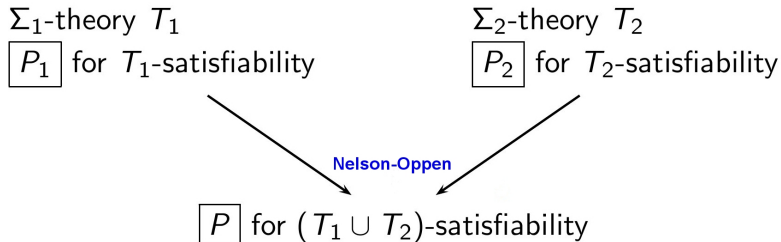
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- ▶ **Today's lecture:** Learn about **Nelson-Oppen method** for constructing decision procedure for combined theory $T_1 \cup T_2$ from individual decision procedures for T_1 and T_2

Nelson-Oppen Overview

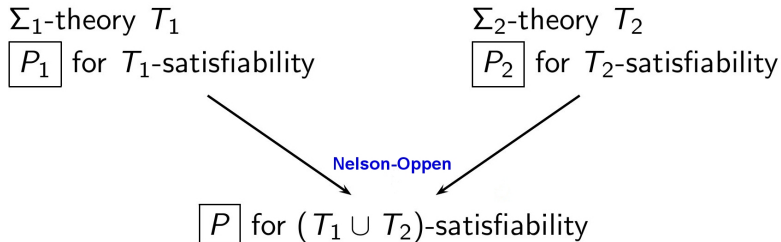


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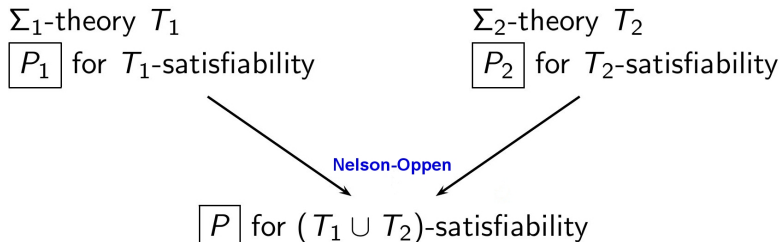
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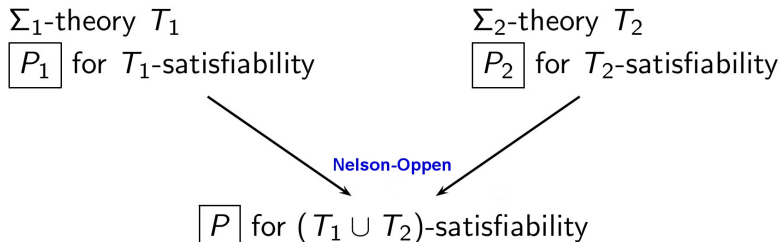
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- ▶ However, Nelson-Open imposes some restrictions on theories that can be combined

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- ▶ Thus, theories with only finite models are not stably infinite.

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- ▶ Hence, theory only has finite models, and is **not** stably infinite

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- ▶ More recent work has also extended Nelson-Oppen to non-stably-infinite theories

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- ▶ But equality propagation is different between **convex** and **non-convex** theories

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- ▶ But since goal is to decide satisfiability, this is good enough

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- ▶ Thus, we can write F as a conjunction of formulas F_1 in T_1 and F_2 in T_2

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- ▶ Thus, formula after purification:

$$\underbrace{x \leq y + 1}_{T_{\mathbb{Q}}} \wedge \underbrace{y = f(x)}_{T_{=}}$$

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- ▶ Is formula purified now? **Yes, finally!**

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Two Phases of Nelson-Oppen

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- ▶ So, need to talk about convex and non-convex theories

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- ▶ Thus, in convex theory, if F implies disjunction of equalities, F also implies at least one of these equalities on its own
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- ▶ Is $T_{\mathbb{Z}}$ convex? **no**
- ▶ Theory of equality $T_{=}$ is **convex**

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- ▶ We'll first talk about Nelson-Oppen method for convex theories, then for non-convex theories

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- ▶ Because F is equisatisfiable to $F_1 \wedge F_2$, which is unsat

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- ▶ Why only equalities? b/c it is the only shared symbol

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- Use Nelson-Oppen to decide sat of following $T_{=} \cup T_{\mathbb{Q}}$ formula:

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- ▶ First literal is now $f(w_1 - w_2) \neq f(z)$; still not pure!

Example

- ▶ Use Nelson-Oppen to decide sat of following $T_{=} \cup T_{\mathbb{Q}}$ formula:

$$f(f(x) - f(y)) \neq f(z) \wedge x \leq y \wedge y + z \leq x \wedge 0 \leq z$$

- ▶ First, we need to purify:

- ▶ Replace $f(x)$ with new variable w_1

- ▶ Replace $f(y)$ with new variable w_2

- ▶ $f(x) - f(y)$ is now replaced with $w_1 - w_2$ and we conjoin

$$w_1 = f(x) \wedge w_2 = f(y)$$

- ▶ First literal is now $f(w_1 - w_2) \neq f(z)$; still not pure!

- ▶ Replace $w_1 - w_2$ with w_3 and add equality $w_3 = w_1 - w_2$

Example, cont

- Purified formula is $F_1 \wedge F_2$ where:

$$F_1 : w_1 = f(x) \wedge w_2 = f(y) \wedge f(w_3) \neq f(z)$$

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- Which variables are shared?

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- ▶ Which variables are shared? **all**

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- ▶ Which variables are shared? **all**
- ▶ Check sat of F_1 . Is it SAT? **yes**
- ▶ Check sat of F_2 . Is it SAT? **yes**
- ▶ Now, for each pair of shared variable x_i, x_j , we query whether F_1 or F_2 imply $x_i = x_j$

Example, cont

$$F_1 : w_1 = f(x) \wedge w_2 = f(y) \wedge f(w_3) \neq f(z)$$

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Example, cont

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Example, cont

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- ▶ Are we done? **no**

Example, cont

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Example, cont

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Example, cont

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- ▶ Since F_1 changed, need to check if it implies any new equality
- ▶ Does it imply a new equality? **yes, $w_1 = w_2$**
- ▶ Now, we add $w_1 = w_2$ to F_2 :

$$F_2 : w_3 = w_1 - w_2 \wedge x \leq y \wedge y + z \leq x \wedge 0 \leq z \wedge w_1 = w_2$$

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- ▶ We recheck sat of F_2 . Is it SAT? **yes**
- ▶ Still not done b/c need to check if F_2 implies any new equalities

Example, cont

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- Consider the query $w_3 = z?$

Example, cont

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- ▶ Consider the query $w_3 = z$?
- ▶ $w_3 = w_1 - w_2$ and $w_1 = w_2$ imply $w_3 = 0$

Example, cont

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Example, cont

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- ▶ Consider the query $w_3 = z$?
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- ▶ Since $x = y$, $y + z \leq x$ implies $z \leq 0$
- ▶ Since $z \leq 0$ and $0 \leq z$, we have $z = 0$
- ▶ Thus, $T_{\mathbb{Q}}$ answer "yes" for query $w_3 = z$

Example, cont

- Now, propagate $w_3 = z$ to F_1 :

$$F_1 : w_1 = f(x) \wedge w_2 = f(y) \wedge f(w_3) \neq f(z) \wedge x = y \wedge w_3 = z$$

Example, cont

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Example, cont

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- ▶ Is this sat?
- ▶ No, because $w_3 = z$ implies $f(w_3) = f(z)$
- ▶ This contradicts $f(w_3) \neq f(z)$
- ▶ Thus, original formula is **UNSAT**

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- ▶ Is this formula SAT? **no**
- ▶ Let's see what happens if we use technique described so far
- ▶ If we purify, we get the following formulas:

$$\begin{aligned} F_1 : & \quad f(x) \neq f(w_1) \wedge f(x) \neq f(w_2) \\ F_2 : & \quad 1 \leq x \wedge x \leq 2 \wedge w_1 = 1 \wedge w_2 = 2 \end{aligned}$$

Example, cont

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- ▶ Is F_2 SAT? **yes**
- ▶ Does F_1 imply a new equality by itself? **no**
- ▶ Does F_2 imply a new equality by itself? **no**
- ▶ Thus technique discussed so far returns sat, although formula is unsat

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- ▶ Thus, it is not enough to query individual equality relations between variables
- ▶ We also have to query and propagate disjunctions of equalities
- ▶ Two questions:

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- ▶ But it doesn't have to imply any single equality on its own
- ▶ Thus, it is not enough to query individual equality relations between variables
- ▶ We also have to query and propagate disjunctions of equalities
- ▶ Two questions:
 1. Which disjunctions do we query?

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- ▶ We also have to query and propagate disjunctions of equalities
- ▶ Two questions:
 1. Which disjunctions do we query?
 2. How do we propagate disjunctions since we are considering disjunction-free formulas?

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- ▶ **Example:** If we have shared variables x, y, z , which queries do we need to issue?

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- ▶ **Example:** If we have shared variables x, y, z , which queries do we need to issue?

$$x = y$$

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$$y = z$$

$$x = y \vee x = z$$

Propagating Disjunctions

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- ▶ Thus, create n subproblems where we propagate $x_i = y_i$ in i 'th subproblem
- ▶ If there is **any** subproblem that is satisfiable, original formula is satisfiable
- ▶ If **every** subproblem is unsatisfiable, then original formula is unsatisfiable

Example

- Consider $T_{=} \cup T_{\mathbb{Z}}$ formula:

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- ▶ Answer to queries (1) and (2) are **no**, but F_2 implies query (3)

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- ▶ Is this satisfiable?
- ▶ **No** because $x = w_1$ implies $f(x) = f(w_1)$

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- ▶ Second subproblem:

$$F_1 : f(x) \neq f(w_1) \wedge f(x) \neq f(w_2) \wedge x = w_2$$

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- ▶ Is this satisfiable?
- ▶ No because $x = w_2$ implies $f(x) = f(w_2)$
- ▶ Since neither subproblem is satisfiable, Nelson-Oppen returns **unsat** for original formula

Example II

- Consider the following $T_{=} \cup T_{\mathbb{Z}}$ formula:

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- ▶ Thus, we now issue new queries such as $x = w_1, x = w_2$, etc
- ▶ Are there any new implied equalities or disjunctions of equalities? **No**
- ▶ Thus, second subproblem is satisfiable
- ▶ Do we need to check third subproblem? **No**
- ▶ Thus, original formula is **satisfiable**

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 2. When we propagate the query, using minimal query creates fewer subproblems

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- ▶ If it is still implied, continue with smaller disjunction
- ▶ Otherwise, restore equality and continue with next one
- ▶ This ensures we find a **minimal disjunction** that is implied
- ▶ This strategy much better than using **any** disjunction that is implied

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- ▶ In non-convex theories:
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 2. If decision procedures for T_1 and T_2 have NP time complexity, combination using Nelson-Oppen also has NP time complexity

Summary

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- ▶ Not a severe restriction because most theories of interest are infinitely stable
- ▶ **Next lecture:** How to decide satisfiability in first-order theories without converting to DNF

Summary

- ▶ Nelson-Oppen method gives a **sound and complete** decision procedure for combination theories
- ▶ However, it only works for quantifier-free theories that are infinitely stable
- ▶ Not a severe restriction because most theories of interest are infinitely stable
- ▶ **Next lecture:** How to decide satisfiability in first-order theories without converting to DNF
- ▶ **Reminder:** homework due next lecture