A non-commutative algorithm for multiplying (7×7) matrices using 250 multiplications

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Abstract

We present a non-commutative algorithm for multiplying (7×7) matrices using 250 multiplications and a non-commutative algorithm for multiplying (9×9) matrices using 520 multiplications. These algorithms are obtained using the same divide-and-conquer technique that could be applied to any suitable matrix sizes.

1 Introduction

The main tool of this note could be summarised in the following proposition:

Proposition 1 — *Denoting by* $\langle u, v, w\rangle$ *the number of multiplications necessary to multiply an* $(u \times v)$ *matrix with an* $(v \times w)$ *matrix to yield an* $(u \times w)$ *product matrix, the following relation holds:*

$$\langle u+v, u+v, u+v\rangle \leq \langle u, u, u\rangle + 3\,\langle u, u, v\rangle + 3\,\langle v, v, u\rangle \text{ when } u > v. \quad (1)$$

For $(u, v) = (4, 3)$, by selecting already known matrix multiplication algorithms and applying this proposition, we obtain a new upper bounds 250 and the explicit corresponding algorithm $(7 \times 7 \times 7\,; 250)$.

In fact, we use the Strassen's matrix multiplication algorithm [13] to *divide* the (7×7) matrix multiplication problem into smaller sub-problems; the use of three Smirnov's rectangular matrix multiplication algorithms [11, 12] allows to *conquer* new upper bounds on the number of necessary non-commutative multiplications.

To illustrate this point, we first present a scheme that evaluate the product $P = N \cdot M$:

$$N = \begin{pmatrix} n_{11} & \cdots & n_{17} \\ \vdots & & \vdots \\ n_{71} & \cdots & n_{77} \end{pmatrix},\ M = \begin{pmatrix} m_{11} & \cdots & m_{17} \\ \vdots & & \vdots \\ m_{71} & \cdots & m_{77} \end{pmatrix},\ P = \begin{pmatrix} p_{11} & \cdots & p_{17} \\ \vdots & & \vdots \\ p_{71} & \cdots & p_{77} \end{pmatrix}, \quad (2)$$

using 250 multiplications. This algorithm improves slightly the previous known upper bound 258 presented in [2] and likely obtained with the same kind of techniques presented in this note.

In the last section of this work, we stress the main limitation of our approach by constructing a (9×9) matrix multiplication algorithm using 520 multiplications (that is only two multiplications less then the corresponding result in [2] but 6 multiplications more then the algorithm $(9 \times 9 \times 9\,;514)$ cited in Table A—that summarise what we know about 286 matrix multiplication algorithms and was obtained automatically during the elaboration of this note—see [10] for a more complete list with all details). This shows that, as the approach presented here is based on the knowledge of fast matrix multiplication algorithms for rectangular matrices, it is limited by the restricted knowledge we have on these algorithms.

2 Divide

For any (2×2) matrices:

$$A = (a_{ij})_{1 \leq i,j \leq 2}, \quad B = (b_{ij})_{1 \leq i,j \leq 2} \quad \text{and} \quad C = (c_{ij})_{1 \leq i,j \leq 2}, \tag{3}$$

V. Strassen shows in [13] that the matrix product $C = A{\cdot}B$ could be computed by performing the following operations:

$$\begin{gathered} t_1 = (a_{11} + a_{22})(b_{11} + b_{22}), \qquad t_2 = (a_{12} - a_{22})(b_{21} + b_{22}), \\ t_3 = (-a_{11} + a_{21})(b_{11} + b_{12}), \quad t_4 = (a_{11} + a_{12})b_{22}, \\ t_5 = a_{11}(b_{12} - b_{22}),\; t_6 = a_{22}(-b_{11} + b_{21}),\; t_7 = (a_{21} + a_{22})b_{11}, \\ \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} t_1 + t_2 - t_4 + t_6 & t_6 + t_7, \\ t_4 + t_5 & t_1 + t_3 + t_5 - t_7 \end{pmatrix}, \end{gathered} \tag{4}$$

in the considered non-necessarily commutative coefficients algebra.

To construct our algorithm, we are going to work with the algebra of (4×4) matrices and thus, we have to adapt our inputs P, N and M (2) to that end. So we rewrite these matrices (2) in the following equivalent form:

$$X = \left(\begin{array}{cccc|ccc} x_{11} & x_{12} & x_{13} & 0 & x_{14} & \cdots & x_{17} \\ x_{21} & x_{22} & x_{23} & 0 & x_{24} & \cdots & x_{27} \\ x_{31} & x_{32} & x_{33} & 0 & x_{34} & \cdots & x_{37} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline x_{41} & x_{42} & x_{43} & 0 & x_{44} & \cdots & x_{47} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ x_{71} & x_{72} & x_{73} & 0 & x_{74} & \cdots & x_{77} \end{array}\right), \quad X \in \{P, M, N\}, \tag{5}$$

in which we have just added a line and a column of zeros. After that padding, the product $P = N \cdot M$ is unchanged.

Notations 2 — *Hence, in the sequel P (resp. M, N) designates the (8×8) matrices defined in (5) and P_{ij} (resp. M_{ij}, N_{ij}) designates (4×4) matrices (e.g. P_{11} stands for the upper left submatrix of P, etc).*

Remark 3 — *The process of peeling the result (removing rows and columns) of our computations might be better understood if we use—even implicitly—the tensor interpretation of matrix multiplication algorithms. Using this framework it* *appears that the bilinear application $\mathcal{B} : \mathbb{K}^{(8\times8)} \times \mathbb{K}^{(8\times8)} \mapsto \mathbb{K}^{(8\times8)}$ with indeterminates N and M that defines the matrix multiplication $\mathcal{B}(N, M) = N \cdot M = P$ is completely equivalent to the trilinear form $\mathbb{K}^{(8\times8)} \times \mathbb{K}^{(8\times8)} \times \mathbb{K}^{(8\times8)} \mapsto \mathbb{K}$ with indeterminates N, M and P defined by $\operatorname{Trace}(N \cdot M \cdot {}^{\intercal}P)$ (e.g. see [6, §4.6.4, page 506], [8, § 2.5.2], [1, § 2.2] or [3] for a complete description of this equivalence).*

Computationally, this equivalence induces that the following relation holds:

$$\begin{aligned}
\operatorname{Trace}(N \cdot M \cdot {}^{\intercal}P) = {} & \operatorname{Trace}((N_{11} + N_{22}) \cdot (M_{11} + M_{22}) \cdot {}^{\intercal}(P_{11} + P_{22})) && \text{(6a)}\\
& + \operatorname{Trace}((N_{12} - N_{22}) \cdot (M_{21} + M_{22}) \cdot {}^{\intercal}P_{11}) && \text{(6b)}\\
& + \operatorname{Trace}((N_{21} - N_{11}) \cdot (M_{11} + M_{12}) \cdot {}^{\intercal}P_{22}) && \text{(6c)}\\
& + \operatorname{Trace}((N_{11} + N_{12}) \cdot M_{22} \cdot {}^{\intercal}(P_{21} - P_{11})) && \text{(6d)}\\
& + \operatorname{Trace}(N_{11} \cdot (M_{12} - M_{22}) \cdot {}^{\intercal}(P_{21} + P_{22})) && \text{(6e)}\\
& + \operatorname{Trace}(N_{22} \cdot (M_{21} - M_{11}) \cdot {}^{\intercal}(P_{11} + P_{12})) && \text{(6f)}\\
& + \operatorname{Trace}((N_{21} + N_{22}) \cdot M_{11} \cdot {}^{\intercal}(P_{12} - P_{22})) && \text{(6g)}
\end{aligned}$$

and as the bilinear application and the trilinear form are equivalent, one could retrieve directly the algorithm from this last form. Our original problem is now divided in 7 lower dimensional subproblems encoded by trilinear forms. In the next section, we enumerate the algorithms used to compute the matrix products (6a-6g).

3 Conquer

The first summand (6a) involves unstructured (4×4) matrix multiplication that could be computed using Strassen's algorithm and could be done with 7^2 multiplications. Before studying the other summands, we emphasise the following trivial remarks:

Remarks 4 — *As we consider some matrices with zero last column and/or row, recall that the product of two (4×4) matrices $(x_{ij})_{1\leq i,j\leq 4}$ and $(y_{ij})_{1\leq i,j\leq 4}$ is equivalent to the product of the:*

- *(4×3) matrix $(x_{ij})_{1\leq i\leq 4, 1\leq j\leq 3}$ with the (3×4) matrix $(y_{ij})_{1\leq i\leq 3, 1\leq, j\leq 4}$ when for $1 \leq i \leq 4$ we have $y_{i4} = 0$ (zero last row);*
- *(3×4) matrix $(x_{ij})_{1\leq i\leq 3, 1\leq j\leq 4}$ with the (4×3) matrix $(y_{ij})_{1\leq i\leq 3, 1\leq, j\leq 4}$ when for $1 \leq j \leq 4$ we have $y_{4j} = 0$ (zero last column).*

Let us now review the matrices involved in the summands (6b-6g).

Facts 5 — *We notice that by construction:*

- *the last row and column of X_{11} are only composed by zeros;*
- *X_{22} is a (4×4) matrix;*
- *the last column of $X_{21} - X_{11}$ is only composed by zeros;*
- *the last line of $X_{11} + X_{12}$ is only composed by zeros;*

- *$X_{12} - X_{22}$ and $X_{21} - X_{22}$ are (4×4) matrices without zero row or column.*

Hence, taking into account Remarks 4 and Facts 5, we have a better description of the sub-problems considered in this section:

Remarks 6 — *The summand:*

- *(6b) involves an (3×4) times (4×3) times (3×3) matrices product;*
- *(6c) involves an (4×3) times (3×4) times (4×4) matrices product;*
- *(6d) involves an (3×4) times (4×4) times (4×3) matrices product;*
- *(6e) involves an (3×3) times (3×4) times (4×3) matrices product;*
- *(6f) involves an (4×4) times (4×3) times (3×4) matrices product;*
- *(6g) involves an (4×3) times (3×3) times (3×4) matrices product.*

Remark 7 — *We also rely our construction on the representation of matrix multiplication algorithm by trilinear forms and the underlying tensor representation (see Remark 3) because, as quoted in [6, § 4.6.4 p. 507]:*

> *"[...], a normal scheme for evaluating an $(m \times n)$ times $(n \times s)$ matrix product implies the existence of a normal scheme to evaluate an $(n \times s)$ times $(s \times m)$ matrix product using the same number of chain multiplications."*

This is exactly what we are using in the sequel. To do so and in order to express complexity of the summands (6b-6g), we (re)introduce more precisely the following notations (already used in Proposition 1):

Notation 8 — *For matrices U, V and W of size $(u \times v), (v \times w)$ and $(w \times u)$, we denote by $\langle u, v, w \rangle$ the known supremum on the multiplication necessary for computing $\mathrm{Trace}\,(U \cdot V \cdot {}^{\intercal}W)$ (that is the number of multiplication used by the best known algorithm allowing to compute $U \cdot V = W$ (a.k.a. tensor rank)).*

Remarks 9 — *Using this notation, we see that Remarks 6 can be restated as follow:*

- *(6b) can be computed using $\langle 3, 4, 3 \rangle$ multiplications;*
- *(6c) can be computed using $\langle 4, 3, 4 \rangle$ multiplications;*
- *(6d) can be computed using $\langle 3, 4, 4 \rangle$ multiplications;*
- *(6e) can be computed using $\langle 3, 3, 4 \rangle$ multiplications;*
- *(6f) can be computed using $\langle 4, 4, 3 \rangle$ multiplications;*
- *(6g) can be computed using $\langle 4, 3, 3 \rangle$ multiplications.*

We are going to see that we need only two algorithms to perform all these computations. In fact, the Remark 7 is a direct consequence of the following Trace properties:

$$\begin{aligned}\mathrm{Trace}\,(U\cdot V\cdot W) = \mathrm{Trace}\,(W\cdot U\cdot V) &= \mathrm{Trace}\,(V\cdot W\cdot U),\\ = \mathrm{Trace}\big({}^{\intercal}(U\cdot V\cdot W)\big) &= \mathrm{Trace}\big({}^{\intercal}W\cdot{}^{\intercal}V\cdot{}^{\intercal}U\big).\end{aligned} \tag{7}$$

To be more precise, Relations (7) imply the following well-known result:

Lemma 10 — *The following relations hold:*

$$\langle u, v, w\rangle = \langle w, u, v\rangle = \langle v, w, u\rangle = \langle v, u, w\rangle = \langle w, v, u\rangle = \langle u, w, v\rangle. \tag{8}$$

This allows us to state that algorithm (6b-6g) requires:

$$\langle 4,4,4\rangle + 3\,\langle 3,3,4\rangle + 3\,\langle 3,4,4\rangle \tag{9}$$

multiplications. As A. V. Smirnov states that $\langle 3,3,4\rangle = 29$ and $\langle 3,4,4\rangle = 38$ in [11, Table 1, № 13 and 21], we conclude that our algorithm required 250 multiplications $(7^2 + 3\cdot 29 + 3\cdot 38)$. Furthermore, Smirnov provides in [12] the explicit description of these algorithms, allowing us to do the same with the first algorithm constructed in this note at url:

http://cristal.univ-lille.fr/~sedoglav/FMM/7x7x7.html.

In the next section, we show how to apply the very same manipulations to the product of two (9×9) matrices.

4 An algorithm for multiplying (9×9) matrices

As in the previous section, we are going to pad our (9×9) matrices:

$$N = (n_{ij})_{1\le i,j\le 9},\quad M = (M_{ij})_{1\le i,j\le 9}\quad\text{and}\quad P = (P_{ij})_{1\le i,j\le 9}, \tag{10}$$

in order to work this time with equivalent (12×12) matrices:

$$Y = \left(\begin{array}{cccccc|ccc} y_{11} & y_{12} & y_{13} & 0 & 0 & 0 & y_{14} & \cdots & y_{19}\\ y_{21} & y_{22} & y_{23} & 0 & 0 & 0 & y_{24} & \cdots & y_{29}\\ y_{31} & y_{32} & y_{33} & 0 & 0 & 0 & y_{34} & \cdots & y_{39}\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\ \hline y_{41} & y_{42} & y_{43} & 0 & 0 & 0 & y_{44} & \cdots & y_{49}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots\\ y_{91} & y_{92} & y_{93} & 0 & 0 & 0 & y_{94} & \cdots & y_{99} \end{array}\right),\quad Y\in\{P,M,N\}, \tag{11}$$

After that padding, the product $P = N\cdot M$ is unchanged.

Notations 11 — *In the sequel P (resp. M, N) designates the* (12×12) *matrices defined in (11) and P_{ij} (resp. M_{ij}, N_{ij}) designates* (6×6) *matrices (e.g. P_{11} stands for the upper left submatrix of P, etc).*

The process described in Section 3 remains—mutatis mutandis—exactly the same and we obtain the following special case of Proposition 1:

Lemma 12 — *With the Notations 8, there exists an algorithm that multiply two* (9×9) *matrices using* $\langle 6,6,6\rangle + 3\,\langle 6,6,3\rangle + 3\,\langle 6,3,3\rangle$ *multiplications.*

Again, we found in [11, Table 1, № 27] that $\langle 3,3,6\rangle = 40$ (an explicit form of this algorithm could be found in the source code of [1] or in the collection [10]). But this time, we do not have found in the literature any *fast* matrix multiplication algorithm for $\langle 6,6,3\rangle$ and we have to provide an algorithm for $\langle 6,6,6\rangle$. Nevertheless, as done implicitly in Section 2, we are going to use again the following constructive result on tensor's Kronecker product.

Lemma 13 — *Given an algorithm computing* $(u_1 \times v_1)$ *times* $(v_1 \times w_1)$ *matrix product using* $\langle u_1, v_1, w_1\rangle$ *multiplications and an algorithm computing* $(u_2 \times v_2)$ *times* $(v_2 \times w_2)$ *matrix product using* $\langle u_2, v_2, w_2\rangle$ *multiplications, one can construct an algorithm computing* $(u_1u_2 \times v_1v_2)$ *by* $(v_1v_2 \times w_1w_2)$ *matrix multiplication using* $\langle u_1, v_1, w_1\rangle \cdot \langle u_2, v_2, w_2\rangle$ *multiplications (a.k.a. the tensor's Kronecker product of the two previous algorithms).*

Hence, as we know that trivially $\langle 1,2,2\rangle = 4$, we conclude that $\langle 6,6,6\rangle$ is equal to $\langle 6,3,3\rangle \cdot \langle 1,2,2\rangle$ (that is 160) and that $\langle 6,6,3\rangle = 80$. So, the algorithm constructed in this section requires 520 multiplications $(4 \cdot 40 + 3 \cdot (40 \cdot 2) + 3 \cdot 40)$.

5 Concluding remarks

The complexity of our algorithm for multiplying (9×9) matrices could likely be improved by finding a *better* algorithm for $\langle 6,6,3\rangle$ and $\langle 6,6,6\rangle$ then those used above (algorithms obtained by tensor Kronecker product are not always optimal as shown by the fact that $\langle 3,3,3\rangle \cdot \langle 3,3,3\rangle$ is equal to 529 while computations summarised in Table A show that $\langle 9,9,9\rangle$ is now 514).

By combining tensor's based description of matrix multiplication algorithms with rectangular algorithms found by numerical computer search (see [1, § 2.3.2] and [11]), it is possible—as already shown in [2]—to improve the theoretical complexity of small size matrix products (see Table A in appendix in which new results obtained with the method presented in this note are in bold face).

The author thinks that some symmetry-based geometrical methods could reduce further the upper bounds presented in this note.

References

[1] Benson, A. R., and Ballard, G. A framework for practical parallel fast matrix multiplication. Tech. rep. arXiv:1409.2908v1 Sept. 2014. Source code at `https://github.com/arbenson/fast-matmul`. [Cited § 3, 4, and 5.]

[2] Drevet, C.-É., Nazrul Islam, M., and Schost, É. Optimization techniques for small matrix multiplication. *Theoretical Computer Science 412*, 22 (May 2011), 2219–2236. https://doi.org/10.1016/j.tcs.2010.12.012. [Cited § 1 and 5.]

[3] Dumas, J.-G., and Pan, V. Y. Fast matrix multiplication and symbolic computation. Tech. rep. arXiv:1612.05766v1, Dec. 2015. [Cited § 3.]

[4] Hopcroft, J. E., and Kerr, L. E. On minimizing the number of multiplication necessary for matrix multiplication. *SIAM Journal on Applied Mathematics 20*, 1 (Jan. 1971). https://doi.org/10.1137/0120004. [Cited § 1.]

[5] Hopcroft, J. E., and Musinski, J. Duality applied to the complexity of matrix multiplications and other bilinear forms. In *5th Annual* ACM *Symposium on the theory of computing* (Apr. 30–May 2 1973), A. V. Aho, A. Borodin, R. L. Constable, R. W. Floyd, M. A. Harrison, R. D. Karp, and R. H. Strong, Eds., pp. 73–87. https://doi.org/10.1137/0202013.

[6] Knuth, D. E. *The Art of Computer Programming. Seminumerical Algorithms*, 3 ed., vol. 2 of *Computer Science and Information Processing*. Addison Wesley, Reading, Mass., 1997. https://doi.org/10.1137/1012065. [Cited § 3 and 7.]

[7] Laderman, J. B. A noncommutative algorithm for multiplying 3 × 3 matrices using 23 multiplications. *Bulletin of the American Mathematical Society 82*, 1 (Jan. 1976), 126–128. https://doi.org/10.1090/S0002-9904-1976-13988-2. [Cited § 1.]

[8] Landsberg, J. M. *Tensors: geometry and applications*, vol. 128 of *Graduate Studies in Mathematics*. American Mathematical Society, 2010. https://doi.org/10.1090/gsm/128. [Cited § 3.]

[9] Sedoglavic, A. A non-commutative algorithm for multiplying 5 × 5 matrices using 99 multiplications. Tech. rep. arXiv:1707.06860v1, arXiv, July 2017. [Cited § 1.]

[10] Sedoglavic, A. Fast matrix multiplication database. http://cristal.univ-lille.fr/~sedoglav/FMM/ [Cited § 1 and 4.]

[11] Smirnov, A. V. The bilinear complexity and practical algorithms for matrix multiplication. *Computational Mathematics and Mathematical Physics 53*, 2 (Dec. 2013), 1781–1795. https://doi.org/10.1134/S0965542513120129. [Cited § 1, 3, 4, 5, and 1.]

[12] Smirnov, A. V. Several bilinear algorithms for matrix multiplication. Tech. rep. Jan. 2017. DOI:10.13140/RG.2.2.30005.06886 [Cited § 1 and 3.]

[13] Strassen, V. Gaussian elimination is not optimal. *Numerische Mathematik 13*, 4 (Aug. 1969), 354–356. https://doi.org/10.1007/BF02165411. [Cited § 1, 2, and 1.]

A Summary of known results

We gather below a summary of some known results up to $\langle 12, 12, 12\rangle$.

$\langle 1, x, y\rangle$	$x \cdot y$	$x \cdot y$	trivial algorithms
$\langle 2, 2, 2\rangle$	7	8	Strassen [13]

$\langle 2,3,3\rangle$	15	18	Hopcroft & Kerr [4]
$\langle 2,3,4\rangle$	20	24	Hopcroft & Kerr [4]
$\langle 2,3,5\rangle$	25	30	Hopcroft & Kerr [4]
$\langle 2,3,6\rangle$	30	36	$\langle 2,3,3\rangle \cdot \langle 1,1,2\rangle$
$\langle 2,3,7\rangle$	35	42	$\langle 2,3,3\rangle + \langle 2,3,4\rangle$
$\langle 2,3,8\rangle$	40	48	$\langle 2,3,4\rangle \cdot \langle 1,1,2\rangle$
$\langle 2,3,9\rangle$	45	54	$\langle 2,3,3\rangle \cdot \langle 1,1,3\rangle$
$\langle 2,3,10\rangle$	50	60	$\langle 2,3,4\rangle + \langle 2,3,6\rangle$
$\langle 2,3,11\rangle$	55	66	$\langle 2,3,7\rangle + \langle 2,3,4\rangle$
$\langle 2,3,12\rangle$	60	72	$\langle 1,1,2\rangle \cdot \langle 2,3,6\rangle$
$\langle 2,4,4\rangle$	26	32	Hopcroft & Kerr [4]
$\langle 2,4,5\rangle$	33	40	Hopcroft & Kerr [4]
$\langle 2,4,6\rangle$	39	48	Hopcroft & Kerr [4]
$\langle 2,4,7\rangle$	46	56	$\langle 2,4,3\rangle + \langle 2,4,4\rangle$
$\langle 2,4,8\rangle$	52	64	$\langle 2,4,4\rangle \cdot \langle 1,1,2\rangle$
$\langle 2,4,9\rangle$	59	72	$\langle 2,4,4\rangle + \langle 2,4,5\rangle$
$\langle 2,4,10\rangle$	65	80	$\langle 2,4,4\rangle + \langle 2,4,6\rangle$
$\langle 2,4,11\rangle$	72	88	$\langle 2,4,7\rangle + \langle 2,4,4\rangle$
$\langle 2,4,12\rangle$	78	96	$\langle 1,1,3\rangle \cdot \langle 2,4,4\rangle$
$\langle 2,5,5\rangle$	40	50	Hopcroft & Kerr [4]
$\langle 2,5,6\rangle$	48	60	Hopcroft & Kerr [4]
$\langle 2,5,7\rangle$	56	70	Hopcroft & Kerr [4]
$\langle 2,5,8\rangle$	64	80	Hopcroft & Kerr [4]
$\langle 2,5,9\rangle$	72	90	Hopcroft & Kerr [4]
$\langle 2,5,10\rangle$	80	100	$\langle 2,5,5\rangle \cdot \langle 1,1,2\rangle$
$\langle 2,5,11\rangle$	88	110	$\langle 2,5,6\rangle + \langle 2,5,5\rangle$
$\langle 2,5,12\rangle$	96	120	$\langle 2,5,6\rangle + \langle 2,5,6\rangle$
$\langle 2,6,6\rangle$	57	72	Hopcroft & Kerr [4]
$\langle 2,6,7\rangle$	67	84	Hopcroft & Kerr [4]
$\langle 2,6,8\rangle$	76	96	Hopcroft & Kerr [4]
$\langle 2,6,9\rangle$	87	108	$\langle 2,6,4\rangle + \langle 2,6,5\rangle$
$\langle 2,6,10\rangle$	95	120	Hopcroft & Kerr [4]
$\langle 2,6,11\rangle$	105	132	$\langle 2,6,5\rangle + \langle 2,6,6\rangle$
$\langle 2,6,12\rangle$	114	144	$\langle 1,1,2\rangle \cdot \langle 2,6,6\rangle$
$\langle 2,7,7\rangle$	77	98	Hopcroft & Kerr [4]
$\langle 2,7,8\rangle$	91	112	$\langle 2,7,3\rangle + \langle 2,7,5\rangle$
$\langle 2,7,9\rangle$	99	126	Hopcroft & Kerr [4]
$\langle 2,7,10\rangle$	110	140	Hopcroft & Kerr [4]
$\langle 2,7,11\rangle$	121	154	Hopcroft & Kerr [4]
$\langle 2,7,12\rangle$	132	168	Hopcroft & Kerr [4]
$\langle 2,8,8\rangle$	100	128	Hopcroft & Kerr [4]
$\langle 2,8,9\rangle$	116	144	$\langle 2,8,4\rangle + \langle 2,8,5\rangle$
$\langle 2,8,10\rangle$	128	160	$\langle 1,1,2\rangle \cdot \langle 2,8,5\rangle$
$\langle 2,8,11\rangle$	140	176	$\langle 2,8,3\rangle + \langle 2,8,8\rangle$
$\langle 2,8,12\rangle$	152	192	$\langle 2,8,4\rangle + \langle 2,8,8\rangle$
$\langle 2,9,9\rangle$	126	162	Hopcroft & Kerr [4]
$\langle 2,9,10\rangle$	144	180	$\langle 1,1,2\rangle \cdot \langle 2,9,5\rangle$
$\langle 2,9,11\rangle$	158	198	$\langle 2,9,2\rangle + \langle 2,9,9\rangle$
$\langle 2,9,12\rangle$	168	216	Hopcroft & Kerr [4]
$\langle 2,10,10\rangle$	155	200	Hopcroft & Kerr [4]

$\langle 2,10,11\rangle$	175	220	$\langle 2,10,1\rangle + \langle 2,10,10\rangle$
$\langle 2,10,12\rangle$	190	240	$\langle 2,10,2\rangle + \langle 2,10,10\rangle$
$\langle 2,11,11\rangle$	187	242	Hopcroft & Kerr [4]
$\langle 2,11,12\rangle$	209	264	$\langle 2,11,1\rangle + \langle 2,11,11\rangle$
$\langle 2,12,12\rangle$	222	288	Hopcroft & Kerr [4]
$\langle 3,3,3\rangle$	23	27	Laderman [7]
$\langle 3,3,4\rangle$	29	36	Smirnov [11]
$\langle 3,3,5\rangle$	36	45	Smirnov [11]
$\langle 3,3,6\rangle$	40	54	Smirnov [11]
$\langle 3,3,7\rangle$	49	63	$\langle 3,3,1\rangle + \langle 3,3,6\rangle$
$\langle 3,3,8\rangle$	55	72	$\langle 3,3,6\rangle + \langle 3,3,2\rangle$
$\langle 3,3,9\rangle$	63	81	$\langle 3,3,3\rangle + \langle 3,3,6\rangle$
$\langle 3,3,10\rangle$	69	90	$\langle 3,3,4\rangle + \langle 3,3,6\rangle$
$\langle 3,3,11\rangle$	76	99	$\langle 3,3,5\rangle + \langle 3,3,6\rangle$
$\langle 3,3,12\rangle$	80	108	$\langle 1,1,2\rangle \cdot \langle 3,3,6\rangle$
$\langle 3,4,4\rangle$	38	48	Smirnov [11]
$\langle 3,4,5\rangle$	48	60	Smirnov [11]
$\langle 3,4,6\rangle$	58	72	$\langle 3,4,3\rangle \cdot \langle 1,1,2\rangle$
$\langle 3,4,7\rangle$	67	84	$\langle 3,4,3\rangle + \langle 3,4,4\rangle$
$\langle 3,4,8\rangle$	76	96	$\langle 3,4,4\rangle \cdot \langle 1,1,2\rangle$
$\langle 3,4,9\rangle$	86	108	$\langle 3,4,4\rangle + \langle 3,4,5\rangle$
$\langle 3,4,10\rangle$	96	120	$\langle 3,4,2\rangle + \langle 3,4,8\rangle$
$\langle 3,4,11\rangle$	105	132	$\langle 3,4,3\rangle + \langle 3,4,8\rangle$
$\langle 3,4,12\rangle$	114	144	$\langle 1,1,3\rangle \cdot \langle 3,4,4\rangle$
$\langle 3,5,5\rangle$	61	75	$\langle 3,5,2\rangle + \langle 3,5,3\rangle$
$\langle 3,5,6\rangle$	70	90	$\langle 3,2,6\rangle + \langle 3,3,6\rangle$
$\langle 3,5,7\rangle$	84	105	$\langle 3,2,7\rangle + \langle 3,3,7\rangle$
$\langle 3,5,8\rangle$	95	120	$\langle 3,5,6\rangle + \langle 3,5,2\rangle$
$\langle 3,5,9\rangle$	106	135	$\langle 3,5,3\rangle + \langle 3,5,6\rangle$
$\langle 3,5,10\rangle$	118	150	$\langle 3,5,4\rangle + \langle 3,5,6\rangle$
$\langle 3,5,11\rangle$	**130**	165	$2\,\langle 2,3,5\rangle + 2\,\langle 3,3,6\rangle$
$\langle 3,5,12\rangle$	**140**	180	$2\,\langle 2,3,6\rangle + 2\,\langle 3,3,6\rangle$
$\langle 3,6,6\rangle$	80	108	$\langle 3,3,6\rangle \cdot \langle 1,2,1\rangle$
$\langle 3,6,7\rangle$	98	126	$\langle 3,3,7\rangle \cdot \langle 1,2,1\rangle$
$\langle 3,6,8\rangle$	110	144	$\langle 1,2,1\rangle \cdot \langle 3,3,8\rangle$
$\langle 3,6,9\rangle$	120	162	$\langle 3,6,3\rangle \cdot \langle 1,1,3\rangle$
$\langle 3,6,10\rangle$	138	180	$\langle 3,6,1\rangle + \langle 3,6,9\rangle$
$\langle 3,6,11\rangle$	150	198	$\langle 3,6,2\rangle + \langle 3,6,9\rangle$
$\langle 3,6,12\rangle$	160	216	$\langle 1,1,2\rangle \cdot \langle 3,6,6\rangle$
$\langle 3,7,7\rangle$	116	147	$\langle 3,7,3\rangle + \langle 3,7,4\rangle$
$\langle 3,7,8\rangle$	131	168	$\langle 3,3,8\rangle + \langle 3,4,8\rangle$
$\langle 3,7,9\rangle$	147	189	$\langle 3,7,3\rangle \cdot \langle 1,1,3\rangle$
$\langle 3,7,10\rangle$	165	210	$\langle 3,7,4\rangle + \langle 3,7,6\rangle$
$\langle 3,7,11\rangle$	180	231	$\langle 3,7,3\rangle + \langle 3,7,8\rangle$
$\langle 3,7,12\rangle$	194	252	$\langle 3,3,12\rangle + \langle 3,4,12\rangle$
$\langle 3,8,8\rangle$	150	192	$\langle 3,8,2\rangle + \langle 3,8,6\rangle$
$\langle 3,8,9\rangle$	165	216	$\langle 1,1,3\rangle \cdot \langle 3,8,3\rangle$
$\langle 3,8,10\rangle$	186	240	$\langle 3,8,4\rangle + \langle 3,8,6\rangle$
$\langle 3,8,11\rangle$	205	264	$\langle 3,8,2\rangle + \langle 3,8,9\rangle$
$\langle 3,8,12\rangle$	220	288	$\langle 1,1,4\rangle \cdot \langle 3,8,3\rangle$

$\langle 3,9,9\rangle$	183	243	$\langle 3,9,3\rangle + \langle 3,9,6\rangle$
$\langle 3,9,10\rangle$	206	270	$\langle 3,9,4\rangle + \langle 3,9,6\rangle$
$\langle 3,9,11\rangle$	226	297	$\langle 3,9,5\rangle + \langle 3,9,6\rangle$
$\langle 3,9,12\rangle$	240	324	$\langle 1,1,2\rangle \cdot \langle 3,9,6\rangle$
$\langle 3,10,10\rangle$	234	300	$\langle 3,10,4\rangle + \langle 3,10,6\rangle$
$\langle 3,10,11\rangle$	255	330	$\langle 3,10,3\rangle + \langle 3,10,8\rangle$
$\langle 3,10,12\rangle$	274	360	$\langle 3,4,12\rangle + \langle 3,6,12\rangle$
$\langle 3,11,11\rangle$	280	363	$\langle 3,11,5\rangle + \langle 3,11,6\rangle$
$\langle 3,11,12\rangle$	300	396	$\langle 3,11,6\rangle + \langle 3,11,6\rangle$
$\langle 3,12,12\rangle$	320	432	$\langle 1,2,2\rangle \cdot \langle 3,6,6\rangle$
$\langle 4,4,4\rangle$	49	64	$\langle 2,2,2\rangle \cdot \langle 2,2,2\rangle$
$\langle 4,4,5\rangle$	64	80	$\langle 4,4,2\rangle + \langle 4,4,3\rangle$
$\langle 4,4,6\rangle$	75	96	$\langle 4,4,4\rangle + \langle 4,4,2\rangle$
$\langle 4,4,7\rangle$	87	112	$\langle 4,4,4\rangle + \langle 4,4,3\rangle$
$\langle 4,4,8\rangle$	98	128	$\langle 1,1,2\rangle \cdot \langle 4,4,4\rangle$
$\langle 4,4,9\rangle$	113	144	$\langle 4,4,2\rangle + \langle 4,4,7\rangle$
$\langle 4,4,10\rangle$	124	160	$\langle 4,4,2\rangle + \langle 4,4,8\rangle$
$\langle 4,4,11\rangle$	136	176	$\langle 4,4,3\rangle + \langle 4,4,8\rangle$
$\langle 4,4,12\rangle$	147	192	$\langle 1,1,3\rangle \cdot \langle 4,4,4\rangle$
$\langle 4,5,5\rangle$	80	100	$\langle 2,5,5\rangle \cdot \langle 2,1,1\rangle$
$\langle 4,5,6\rangle$	**93**	120	$3\,\langle 2,2,3\rangle + 4\,\langle 2,3,3\rangle$
$\langle 4,5,7\rangle$	**109**	140	$\langle 2,2,3\rangle + 2\,\langle 2,2,4\rangle + 2\,\langle 2,3,3\rangle + 2\,\langle 2,3,4\rangle$
$\langle 4,5,8\rangle$	**122**	160	$3\,\langle 2,2,4\rangle + 4\,\langle 2,3,4\rangle$
$\langle 4,5,9\rangle$	**140**	180	$\langle 2,2,4\rangle + 2\,\langle 2,2,5\rangle + 2\,\langle 2,3,4\rangle + 2\,\langle 2,3,5\rangle$
$\langle 4,5,10\rangle$	**154**	200	$3\,\langle 2,2,5\rangle + 4\,\langle 2,3,5\rangle$
$\langle 4,5,11\rangle$	**170**	220	$\langle 2,2,5\rangle + 2\,\langle 2,2,6\rangle + 2\,\langle 2,3,5\rangle + 2\,\langle 2,3,6\rangle$
$\langle 4,5,12\rangle$	**183**	240	$3\,\langle 2,2,6\rangle + 4\,\langle 2,3,6\rangle$
$\langle 4,6,6\rangle$	105	144	$\langle 2,2,2\rangle \cdot \langle 2,3,3\rangle$
$\langle 4,6,7\rangle$	**125**	168	$3\,\langle 2,3,3\rangle + 4\,\langle 2,3,4\rangle$
$\langle 4,6,8\rangle$	140	192	$\langle 2,2,2\rangle \cdot \langle 2,3,4\rangle$
$\langle 4,6,9\rangle$	**160**	216	$3\,\langle 2,3,4\rangle + 4\,\langle 2,3,5\rangle$
$\langle 4,6,10\rangle$	175	240	$\langle 2,2,2\rangle \cdot \langle 2,3,5\rangle$
$\langle 4,6,11\rangle$	198	264	$\langle 4,6,3\rangle + \langle 4,6,8\rangle$
$\langle 4,6,12\rangle$	210	288	$\langle 1,1,2\rangle \cdot \langle 4,6,6\rangle$
$\langle 4,7,7\rangle$	**147**	196	$\langle 2,3,3\rangle + 2\,\langle 2,4,4\rangle + 4\,\langle 2,3,4\rangle$
$\langle 4,7,8\rangle$	**164**	224	$3\,\langle 2,3,4\rangle + 4\,\langle 2,4,4\rangle$
$\langle 4,7,9\rangle$	**188**	252	$\langle 2,3,4\rangle + 2\,\langle 2,3,5\rangle + 2\,\langle 2,4,4\rangle + 2\,\langle 2,4,5\rangle$
$\langle 4,7,10\rangle$	**207**	280	$3\,\langle 2,3,5\rangle + 4\,\langle 2,4,5\rangle$
$\langle 4,7,11\rangle$	**229**	308	$\langle 2,3,5\rangle + 2\,\langle 2,3,6\rangle + 2\,\langle 2,4,5\rangle + 2\,\langle 2,4,6\rangle$
$\langle 4,7,12\rangle$	**246**	336	$3\,\langle 2,3,6\rangle + 4\,\langle 2,4,6\rangle$
$\langle 4,8,8\rangle$	182	256	$\langle 2,2,2\rangle \cdot \langle 2,4,4\rangle$
$\langle 4,8,9\rangle$	**210**	288	$3\,\langle 2,4,4\rangle + 4\,\langle 2,4,5\rangle$
$\langle 4,8,10\rangle$	231	320	$\langle 2,2,2\rangle \cdot \langle 2,4,5\rangle$
$\langle 4,8,11\rangle$	**255**	352	$3\,\langle 2,4,5\rangle + 4\,\langle 2,4,6\rangle$
$\langle 4,8,12\rangle$	273	384	$\langle 2,2,2\rangle \cdot \langle 2,4,6\rangle$
$\langle 4,9,9\rangle$	225	324	$\langle 2,3,3\rangle \cdot \langle 2,3,3\rangle$
$\langle 4,9,10\rangle$	**259**	360	$3\,\langle 2,4,5\rangle + 4\,\langle 2,5,5\rangle$
$\langle 4,9,11\rangle$	284	396	$\langle 4,9,2\rangle + \langle 4,9,9\rangle$
$\langle 4,9,12\rangle$	300	432	$\langle 2,3,3\rangle \cdot \langle 2,3,4\rangle$
$\langle 4,10,10\rangle$	280	400	$\langle 2,2,2\rangle \cdot \langle 2,5,5\rangle$

$\langle 4, 10, 11\rangle$	320	440	$\langle 4, 10, 1\rangle + \langle 4, 10, 10\rangle$
$\langle 4, 10, 12\rangle$	336	480	$\langle 2, 2, 2\rangle \cdot \langle 2, 5, 6\rangle$
$\langle 4, 11, 11\rangle$	**346**	484	$\langle 2, 5, 5\rangle + 2\,\langle 2, 6, 6\rangle + 4\,\langle 2, 5, 6\rangle$
$\langle 4, 11, 12\rangle$	**372**	528	$3\,\langle 2, 5, 6\rangle + 4\,\langle 2, 6, 6\rangle$
$\langle 4, 12, 12\rangle$	390	576	$\langle 2, 3, 3\rangle \cdot \langle 2, 4, 4\rangle$
$\langle 5, 5, 5\rangle$	99	125	Sedoglavic [9]
$\langle 5, 5, 6\rangle$	**117**	150	$\langle 2, 2, 3\rangle + 2\,\langle 3, 3, 3\rangle + 4\,\langle 2, 3, 3\rangle$
$\langle 5, 5, 7\rangle$	**136**	175	$\langle 2, 2, 4\rangle + \langle 3, 3, 3\rangle + \langle 3, 3, 4\rangle + 2\,\langle 2, 3, 3\rangle + 2\,\langle 2, 3, 4\rangle$
$\langle 5, 5, 8\rangle$	**152**	200	$\langle 2, 2, 4\rangle + 2\,\langle 3, 3, 4\rangle + 4\,\langle 2, 3, 4\rangle$
$\langle 5, 5, 9\rangle$	**173**	225	$\langle 2, 2, 5\rangle + \langle 3, 3, 4\rangle + \langle 3, 3, 5\rangle + 2\,\langle 2, 3, 4\rangle + 2\,\langle 2, 3, 5\rangle$
$\langle 5, 5, 10\rangle$	**190**	250	$\langle 2, 2, 5\rangle + 2\,\langle 3, 3, 5\rangle + 4\,\langle 2, 3, 5\rangle$
$\langle 5, 5, 11\rangle$	**206**	275	$\langle 2, 2, 6\rangle + \langle 2, 3, 6\rangle + 2\,\langle 3, 3, 6\rangle + 3\,\langle 2, 3, 5\rangle$
$\langle 5, 5, 12\rangle$	**221**	300	$\langle 2, 2, 6\rangle + 2\,\langle 3, 3, 6\rangle + 4\,\langle 2, 3, 6\rangle$
$\langle 5, 6, 6\rangle$	**137**	180	$3\,\langle 2, 3, 3\rangle + 4\,\langle 3, 3, 3\rangle$
$\langle 5, 6, 7\rangle$	**159**	210	$\langle 2, 3, 3\rangle + 2\,\langle 2, 3, 4\rangle + 2\,\langle 3, 3, 3\rangle + 2\,\langle 3, 3, 4\rangle$
$\langle 5, 6, 8\rangle$	**176**	240	$3\,\langle 2, 3, 4\rangle + 4\,\langle 3, 3, 4\rangle$
$\langle 5, 6, 9\rangle$	**200**	270	$\langle 2, 3, 4\rangle + 2\,\langle 2, 3, 5\rangle + 2\,\langle 3, 3, 4\rangle + 2\,\langle 3, 3, 5\rangle$
$\langle 5, 6, 10\rangle$	**218**	300	$\langle 2, 3, 4\rangle + 2\,\langle 2, 3, 6\rangle + 2\,\langle 3, 3, 4\rangle + 2\,\langle 3, 3, 6\rangle$
$\langle 5, 6, 11\rangle$	**236**	330	$\langle 2, 3, 6\rangle + \langle 3, 3, 5\rangle + 2\,\langle 2, 3, 5\rangle + 3\,\langle 3, 3, 6\rangle$
$\langle 5, 6, 12\rangle$	**250**	360	$3\,\langle 2, 3, 6\rangle + 4\,\langle 3, 3, 6\rangle$
$\langle 5, 7, 7\rangle$	**185**	245	$\langle 2, 4, 4\rangle + \langle 3, 3, 3\rangle + \langle 3, 4, 4\rangle + 2\,\langle 2, 3, 4\rangle + 2\,\langle 3, 3, 4\rangle$
$\langle 5, 7, 8\rangle$	**206**	280	$\langle 2, 3, 4\rangle + 2\,\langle 2, 4, 4\rangle + 2\,\langle 3, 3, 4\rangle + 2\,\langle 3, 4, 4\rangle$
$\langle 5, 7, 9\rangle$	**235**	315	$\langle 2, 3, 5\rangle + \langle 2, 4, 4\rangle + \langle 2, 4, 5\rangle + \langle 3, 3, 4\rangle + \langle 3, 3, 5\rangle + \langle 3, 4, 4\rangle + \langle 3, 4, 5\rangle$
$\langle 5, 7, 10\rangle$	**259**	350	$\langle 2, 3, 5\rangle + 2\,\langle 2, 4, 5\rangle + 2\,\langle 3, 3, 5\rangle + 2\,\langle 3, 4, 5\rangle$
$\langle 5, 7, 11\rangle$	**283**	385	$\langle 2, 3, 5\rangle + \langle 2, 4, 5\rangle + \langle 2, 4, 6\rangle + \langle 3, 4, 5\rangle + \langle 3, 4, 6\rangle + 2\,\langle 3, 3, 6\rangle$
$\langle 5, 7, 12\rangle$	**304**	420	$\langle 2, 3, 6\rangle + 2\,\langle 2, 4, 6\rangle + 2\,\langle 3, 3, 6\rangle + 2\,\langle 3, 4, 6\rangle$
$\langle 5, 8, 8\rangle$	**230**	320	$3\,\langle 2, 4, 4\rangle + 4\,\langle 3, 4, 4\rangle$
$\langle 5, 8, 9\rangle$	**264**	360	$\langle 2, 4, 4\rangle + 2\,\langle 2, 4, 5\rangle + 2\,\langle 3, 4, 4\rangle + 2\,\langle 3, 4, 5\rangle$
$\langle 5, 8, 10\rangle$	**291**	400	$3\,\langle 2, 4, 5\rangle + 4\,\langle 3, 4, 5\rangle$
$\langle 5, 8, 11\rangle$	**323**	440	$\langle 2, 4, 5\rangle + 2\,\langle 2, 4, 6\rangle + 2\,\langle 3, 4, 5\rangle + 2\,\langle 3, 4, 6\rangle$
$\langle 5, 8, 12\rangle$	346	480	$\langle 5, 2, 12\rangle + \langle 5, 6, 12\rangle$
$\langle 5, 9, 9\rangle$	**300**	405	$\langle 2, 6, 6\rangle + \langle 3, 3, 3\rangle + \langle 3, 6, 6\rangle + 2\,\langle 2, 3, 6\rangle + 2\,\langle 3, 3, 6\rangle$
$\langle 5, 9, 10\rangle$	**331**	450	$\langle 2, 4, 5\rangle + 2\,\langle 2, 5, 5\rangle + 2\,\langle 3, 4, 5\rangle + 2\,\langle 3, 5, 5\rangle$
$\langle 5, 9, 11\rangle$	**360**	495	$\langle 2, 3, 5\rangle + \langle 2, 5, 6\rangle + \langle 2, 6, 6\rangle + \langle 3, 5, 6\rangle + \langle 3, 6, 6\rangle + 2\,\langle 3, 3, 6\rangle$
$\langle 5, 9, 12\rangle$	**384**	540	$\langle 2, 3, 6\rangle + 2\,\langle 2, 6, 6\rangle + 2\,\langle 3, 3, 6\rangle + 2\,\langle 3, 6, 6\rangle$
$\langle 5, 10, 10\rangle$	**364**	500	$3\,\langle 2, 5, 5\rangle + 4\,\langle 3, 5, 5\rangle$
$\langle 5, 10, 11\rangle$	**398**	550	$\langle 2, 5, 6\rangle + \langle 3, 5, 5\rangle + 2\,\langle 2, 5, 5\rangle + 3\,\langle 3, 5, 6\rangle$
$\langle 5, 10, 12\rangle$	**424**	600	$3\,\langle 2, 5, 6\rangle + 4\,\langle 3, 5, 6\rangle$
$\langle 5, 11, 11\rangle$	**434**	605	$\langle 2, 6, 6\rangle + \langle 3, 5, 5\rangle + \langle 3, 6, 6\rangle + 2\,\langle 2, 5, 6\rangle + 2\,\langle 3, 5, 6\rangle$
$\langle 5, 11, 12\rangle$	**462**	660	$\langle 2, 5, 6\rangle + 2\,\langle 2, 6, 6\rangle + 2\,\langle 3, 5, 6\rangle + 2\,\langle 3, 6, 6\rangle$
$\langle 5, 12, 12\rangle$	**491**	720	$3\,\langle 2, 6, 6\rangle + 4\,\langle 3, 6, 6\rangle$
$\langle 6, 6, 6\rangle$	160	216	$\langle 3, 3, 6\rangle \cdot \langle 2, 2, 1\rangle$
$\langle 6, 6, 7\rangle$	**185**	252	$3\,\langle 3, 3, 3\rangle + 4\,\langle 3, 3, 4\rangle$
$\langle 6, 6, 8\rangle$	203	288	$\langle 2, 2, 2\rangle \cdot \langle 3, 3, 4\rangle$
$\langle 6, 6, 9\rangle$	225	324	$\langle 2, 3, 3\rangle \cdot \langle 3, 2, 3\rangle$
$\langle 6, 6, 10\rangle$	**247**	360	$3\,\langle 3, 3, 4\rangle + 4\,\langle 3, 3, 6\rangle$
$\langle 6, 6, 11\rangle$	**268**	396	$3\,\langle 3, 3, 5\rangle + 4\,\langle 3, 3, 6\rangle$
$\langle 6, 6, 12\rangle$	280	432	$\langle 2, 2, 2\rangle \cdot \langle 3, 3, 6\rangle$
$\langle 6, 7, 7\rangle$	**215**	294	$\langle 3, 3, 3\rangle + 2\,\langle 3, 4, 4\rangle + 4\,\langle 3, 3, 4\rangle$
$\langle 6, 7, 8\rangle$	**239**	336	$3\,\langle 3, 3, 4\rangle + 4\,\langle 3, 4, 4\rangle$

$\langle 6,7,9\rangle$	**273**	378	$\langle 3,3,4\rangle + 2\,\langle 3,3,5\rangle + 2\,\langle 3,4,4\rangle + 2\,\langle 3,4,5\rangle$
$\langle 6,7,10\rangle$	**300**	420	$3\,\langle 3,3,5\rangle + 4\,\langle 3,4,5\rangle$
$\langle 6,7,11\rangle$	**328**	462	$\langle 3,3,5\rangle + 2\,\langle 3,3,6\rangle + 2\,\langle 3,4,5\rangle + 2\,\langle 3,4,6\rangle$
$\langle 6,7,12\rangle$	**352**	504	$3\,\langle 3,3,6\rangle + 4\,\langle 3,4,6\rangle$
$\langle 6,8,8\rangle$	266	384	$\langle 2,2,2\rangle \cdot \langle 3,4,4\rangle$
$\langle 6,8,9\rangle$	300	432	$\langle 2,4,3\rangle \cdot \langle 3,2,3\rangle$
$\langle 6,8,10\rangle$	336	480	$\langle 2,2,2\rangle \cdot \langle 3,4,5\rangle$
$\langle 6,8,11\rangle$	373	528	$\langle 6,2,11\rangle + \langle 6,6,11\rangle$
$\langle 6,8,12\rangle$	390	576	$\langle 2,4,4\rangle \cdot \langle 3,2,3\rangle$
$\langle 6,9,9\rangle$	**343**	486	$\langle 3,3,3\rangle + 2\,\langle 3,6,6\rangle + 4\,\langle 3,3,6\rangle$
$\langle 6,9,10\rangle$	375	540	$\langle 2,3,5\rangle \cdot \langle 3,3,2\rangle$
$\langle 6,9,11\rangle$	**416**	594	$\langle 3,3,5\rangle + 2\,\langle 3,3,6\rangle + 2\,\langle 3,5,6\rangle + 2\,\langle 3,6,6\rangle$
$\langle 6,9,12\rangle$	435	648	$\langle 2,3,3\rangle \cdot \langle 3,3,4\rangle$
$\langle 6,10,10\rangle$	422	600	$\langle 6,10,4\rangle + \langle 6,10,6\rangle$
$\langle 6,10,11\rangle$	**463**	660	$3\,\langle 3,5,5\rangle + 4\,\langle 3,5,6\rangle$
$\langle 6,10,12\rangle$	490	720	$\langle 2,2,2\rangle \cdot \langle 3,5,6\rangle$
$\langle 6,11,11\rangle$	**501**	726	$\langle 3,5,5\rangle + 2\,\langle 3,6,6\rangle + 4\,\langle 3,5,6\rangle$
$\langle 6,11,12\rangle$	530	792	$\langle 6,5,12\rangle + \langle 6,6,12\rangle$
$\langle 6,12,12\rangle$	560	864	$\langle 2,2,2\rangle \cdot \langle 3,6,6\rangle$
$\langle 7,7,7\rangle$	**250**	343	$\langle 4,4,4\rangle + 3\,\langle 3,3,4\rangle + 3\,\langle 3,4,4\rangle$
$\langle 7,7,8\rangle$	**279**	392	$\langle 3,3,4\rangle + 2\,\langle 4,4,4\rangle + 4\,\langle 3,4,4\rangle$
$\langle 7,7,9\rangle$	**321**	441	$\langle 3,3,5\rangle + \langle 4,4,4\rangle + \langle 4,4,5\rangle + 2\,\langle 3,4,4\rangle + 2\,\langle 3,4,5\rangle$
$\langle 7,7,10\rangle$	**353**	490	$\langle 4,4,4\rangle + 2\,\langle 3,3,6\rangle + 2\,\langle 3,4,4\rangle + 2\,\langle 4,4,6\rangle$
$\langle 7,7,11\rangle$	**388**	539	$\langle 4,4,5\rangle + 2\,\langle 3,3,6\rangle + 2\,\langle 3,4,5\rangle + 2\,\langle 4,4,6\rangle$
$\langle 7,7,12\rangle$	**419**	588	$2\,\langle 3,3,6\rangle + 2\,\langle 3,4,6\rangle + 3\,\langle 4,4,6\rangle$
$\langle 7,8,8\rangle$	**310**	448	$3\,\langle 3,4,4\rangle + 4\,\langle 4,4,4\rangle$
$\langle 7,8,9\rangle$	**360**	504	$\langle 3,4,4\rangle + 2\,\langle 3,4,5\rangle + 2\,\langle 4,4,4\rangle + 2\,\langle 4,4,5\rangle$
$\langle 7,8,10\rangle$	401	560	$\langle 7,8,2\rangle + \langle 7,8,8\rangle$
$\langle 7,8,11\rangle$	441	616	$\langle 7,8,3\rangle + \langle 7,8,8\rangle$
$\langle 7,8,12\rangle$	474	672	$\langle 7,8,4\rangle + \langle 7,8,8\rangle$
$\langle 7,9,9\rangle$	408	567	$\langle 3,9,9\rangle + \langle 4,9,9\rangle$
$\langle 7,9,10\rangle$	**454**	630	$\langle 3,3,6\rangle + \langle 3,4,4\rangle + \langle 3,6,6\rangle + \langle 4,4,6\rangle + \langle 4,6,6\rangle + 2\,\langle 3,4,6\rangle$
$\langle 7,9,11\rangle$	**494**	693	$\langle 3,3,6\rangle + \langle 3,4,5\rangle + \langle 3,4,6\rangle + \langle 3,5,6\rangle + \langle 3,6,6\rangle + \langle 4,5,6\rangle + \langle 4,6,6\rangle$
$\langle 7,9,12\rangle$	**526**	756	$\langle 3,3,6\rangle + 2\,\langle 3,4,6\rangle + 2\,\langle 3,6,6\rangle + 2\,\langle 4,6,6\rangle$
$\langle 7,10,10\rangle$	**500**	700	$\langle 3,6,6\rangle + \langle 4,4,4\rangle + \langle 4,6,6\rangle + 2\,\langle 3,4,6\rangle + 2\,\langle 4,4,6\rangle$
$\langle 7,10,11\rangle$	**545**	770	$\langle 3,4,6\rangle + \langle 3,5,6\rangle + \langle 3,6,6\rangle + \langle 4,4,5\rangle + \langle 4,4,6\rangle + \langle 4,5,6\rangle + \langle 4,6,6\rangle$
$\langle 7,10,12\rangle$	**578**	840	$\langle 3,4,6\rangle + 2\,\langle 3,6,6\rangle + 2\,\langle 4,4,6\rangle + 2\,\langle 4,6,6\rangle$
$\langle 7,11,11\rangle$	**590**	847	$\langle 3,6,6\rangle + 2\,\langle 3,5,6\rangle + 2\,\langle 4,5,5\rangle + 2\,\langle 4,6,6\rangle$
$\langle 7,11,12\rangle$	**626**	924	$\langle 3,5,6\rangle + 2\,\langle 3,6,6\rangle + 2\,\langle 4,5,6\rangle + 2\,\langle 4,6,6\rangle$
$\langle 7,12,12\rangle$	**660**	1008	$3\,\langle 3,6,6\rangle + 4\,\langle 4,6,6\rangle$
$\langle 8,8,8\rangle$	343	512	$\langle 2,2,2\rangle \cdot \langle 4,4,4\rangle$
$\langle 8,8,9\rangle$	400	576	$\langle 2,4,3\rangle \cdot \langle 4,2,3\rangle$
$\langle 8,8,10\rangle$	443	640	$\langle 8,8,2\rangle + \langle 8,8,8\rangle$
$\langle 8,8,11\rangle$	492	704	$\langle 8,8,4\rangle + \langle 8,8,7\rangle$
$\langle 8,8,12\rangle$	520	768	$\langle 2,4,3\rangle \cdot \langle 4,2,4\rangle$
$\langle 8,9,9\rangle$	435	648	$\langle 2,3,3\rangle \cdot \langle 4,3,3\rangle$
$\langle 8,9,10\rangle$	500	720	$\langle 2,3,5\rangle \cdot \langle 4,3,2\rangle$
$\langle 8,9,11\rangle$	551	792	$\langle 8,9,2\rangle + \langle 8,9,9\rangle$
$\langle 8,9,12\rangle$	570	864	$\langle 2,3,3\rangle \cdot \langle 4,3,4\rangle$
$\langle 8,10,10\rangle$	**559**	800	$\langle 4,4,4\rangle + 2\,\langle 4,6,6\rangle + 4\,\langle 4,4,6\rangle$

$\langle 8,10,11\rangle$	**610**	880	$\langle 4,4,5\rangle + 2\,\langle 4,4,6\rangle + 2\,\langle 4,5,6\rangle + 2\,\langle 4,6,6\rangle$
$\langle 8,10,12\rangle$	**645**	960	$3\,\langle 4,4,6\rangle + 4\,\langle 4,6,6\rangle$
$\langle 8,11,11\rangle$	**661**	968	$2\,\langle 4,5,5\rangle + 2\,\langle 4,5,6\rangle + 3\,\langle 4,6,6\rangle$
$\langle 8,11,12\rangle$	**699**	1056	$3\,\langle 4,5,6\rangle + 4\,\langle 4,6,6\rangle$
$\langle 8,12,12\rangle$	735	1152	$\langle 2,2,2\rangle \cdot \langle 4,6,6\rangle$
$\langle 9,9,9\rangle$	514	729	$\mathrm{Proj}([[0,0],[2]],\langle 9,9,10\rangle)$
$\langle 9,9,10\rangle$	540	810	$\langle 3,3,2\rangle \cdot \langle 3,3,5\rangle$
$\langle 9,9,11\rangle$	600	891	$\mathrm{Proj}([[0,0],[12]],\langle 9,9,12\rangle)$
$\langle 9,9,12\rangle$	600	972	$\langle 3,3,2\rangle \cdot \langle 3,3,6\rangle$
$\langle 9,10,10\rangle$	625	900	$\langle 3,2,5\rangle \cdot \langle 3,5,2\rangle$
$\langle 9,10,11\rangle$	684	990	$\langle 9,10,2\rangle + \langle 9,10,9\rangle$
$\langle 9,10,12\rangle$	708	1080	$\langle 9,1,12\rangle + \langle 9,9,12\rangle$
$\langle 9,11,11\rangle$	758	1089	$\langle 9,11,2\rangle + \langle 9,11,9\rangle$
$\langle 9,11,12\rangle$	768	1188	$\langle 9,2,12\rangle + \langle 9,9,12\rangle$
$\langle 9,12,12\rangle$	800	1296	$\langle 3,2,4\rangle \cdot \langle 3,6,3\rangle$
$\langle 10,10,10\rangle$	693	1000	$\langle 2,2,2\rangle \cdot \langle 5,5,5\rangle$
$\langle 10,10,11\rangle$	**765**	1100	$3\,\langle 5,5,5\rangle + 4\,\langle 5,5,6\rangle$
$\langle 10,10,12\rangle$	**815**	1200	$\langle 4,4,6\rangle + 2\,\langle 6,6,6\rangle + 4\,\langle 4,6,6\rangle$
$\langle 10,11,11\rangle$	**841**	1210	$\langle 5,5,5\rangle + 2\,\langle 5,6,6\rangle + 4\,\langle 5,5,6\rangle$
$\langle 10,11,12\rangle$	898	1320	$\langle 10,2,12\rangle + \langle 10,9,12\rangle$
$\langle 10,12,12\rangle$	936	1440	$\langle 2,4,4\rangle \cdot \langle 5,3,3\rangle$
$\langle 11,11,11\rangle$	**922**	1331	$\langle 6,6,6\rangle + 3\,\langle 5,5,6\rangle + 3\,\langle 5,6,6\rangle$
$\langle 11,11,12\rangle$	977	1452	$\langle 2,11,12\rangle + \langle 9,11,12\rangle$
$\langle 11,12,12\rangle$	1022	1584	$\langle 2,12,12\rangle + \langle 9,12,12\rangle$
$\langle 12,12,12\rangle$	1040	1728	$\langle 2,4,4\rangle \cdot \langle 6,3,3\rangle$
$\langle 15,15,15\rangle$	2103	3375	$\langle 15,15,3\rangle + \langle 15,15,12\rangle$
$\langle 18,18,18\rangle$	3200	5832	$\langle 3,3,6\rangle \cdot \langle 6,6,3\rangle$
$\langle 21,21,21\rangle$	5240	9261	$\langle 12,12,12\rangle + 3\,\langle 9,9,12\rangle + 3\,\langle 9,12,12\rangle$