Упражнения по исчислениям предикатов

Натуральное исчисление

Аксиомы:

$\mathfrak{A}: \quad \Gamma, A \vdash A$

$\mathfrak{A}_\vee: \quad \Gamma \vdash A \vee \neg A$

Правила вывода:

$$R^+_\&: \frac{\Gamma \vdash A;\ \Gamma \vdash B}{\Gamma \vdash A \,\&\, B} \qquad R^{-1}_\&: \frac{\Gamma \vdash A \,\&\, B}{\Gamma \vdash A} \qquad R^{-2}_\&: \frac{\Gamma \vdash A \,\&\, B}{\Gamma \vdash B} \qquad R_m: \frac{\Gamma \vdash A}{\Gamma \cup \Delta \vdash A}$$

$$R^{+1}_\vee: \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad R^{+2}_\vee: \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \qquad R^-_\vee: \frac{\Gamma \vdash A \vee B;\ \Gamma, A \vdash C;\ \Gamma, B \vdash C}{\Gamma \vdash C}$$

$$R^+_\to: \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \qquad R^-_\to: \frac{\Gamma \vdash A;\ \Gamma \vdash A \to B}{\Gamma \vdash B} \qquad R^+_\neg: \frac{\Gamma, A \vdash B;\ \Gamma, A \vdash \neg B}{\Gamma \vdash \neg A} \qquad R^-_\neg: \frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$$

$R^+_\forall: \quad \dfrac{\Gamma \vdash A}{\Gamma \vdash \forall \mathtt{x}\ A}$ (x не является свободной переменной формул Γ)

$R^-_\forall: \quad \dfrac{\Gamma \vdash \forall \mathtt{x}\ A}{\Gamma \vdash A\{\mathtt{x}/t\}}$ (переменная x свободна для терма t в формуле A)

$R^+_\exists: \quad \dfrac{\Gamma \vdash A\{\mathtt{x}/t\}}{\Gamma \vdash \exists \mathtt{x}\ A}$ (переменная x свободна для терма t в формуле A)

$R^-_\exists: \quad \dfrac{\Gamma \vdash \exists \mathtt{x}\ A;\ \Gamma, A\{\mathtt{x}/\mathtt{y}\} \vdash B}{\Gamma \vdash B}$ (y не содержится в формулах из $\Gamma \cup \{A, B\}$)

Упражнение 1

Предложить натуральный вывод формулы φ или вывод секвенции $\vdash \varphi$ (A, B, C — произвольные формулы):

1. $A \,\&\, \neg A \to B$
2. $\neg A \to (A \to B)$
3. $(A \to B) \to \neg A \vee B$
4. $\neg A \vee B \to (A \to B)$
5. $A \,\&\, B \to B \,\&\, A$
6. $A \vee B \to B \vee A$
7. $A \,\&(B \vee C) \to A \,\&\, B \vee A \,\&\, C$
8. $A \,\&\, B \vee A \,\&\, C \to A \,\&(B \vee C)$
9. $\neg(A \vee B) \to \neg A \,\&\, \neg B$
10. $\neg A \,\&\, \neg B \to \neg(A \vee B)$
11. $\neg(A \,\&\, B) \to \neg A \vee \neg B$
12. $\neg A \vee \neg B \to \neg(A \,\&\, B)$
13. $\forall \mathtt{x}\ \mathrm{P}(\mathtt{x}) \to \mathrm{P}(\mathbf{c})$
14. $\mathrm{P}(\mathbf{c}) \,\&\, \forall \mathtt{x}\ (\mathrm{P}(\mathtt{x}) \to \mathrm{P}(\mathbf{f}(\mathtt{x}))) \to \mathrm{P}(\mathbf{f}(\mathbf{f}(\mathbf{f}(\mathbf{c}))))$
15. $\forall \mathtt{x}\ \mathrm{P}(\mathtt{x}, \mathtt{x}) \to \forall \mathtt{x}\ \exists \mathtt{y}\ \mathrm{P}(\mathtt{x}, \mathtt{y})$
16. $\exists \mathtt{y}\ \forall \mathtt{x}\ \mathrm{P}(\mathtt{x}, \mathtt{y}) \to \forall \mathtt{x}\ \exists \mathtt{y}\ \mathrm{P}(\mathtt{x}, \mathtt{y})$
17. $\forall \mathtt{x}\ (\mathrm{P}(\mathtt{x}) \to \mathrm{Q}(\mathtt{x})) \to (\forall \mathtt{x}\ \mathrm{P}(\mathtt{x}) \to \forall \mathtt{x}\ \mathrm{Q}(\mathtt{x}))$
18. $\forall \mathtt{x}\ (\mathrm{P}(\mathtt{x}) \,\&\, \mathrm{Q}(\mathtt{x})) \to \forall \mathtt{x}\ \mathrm{P}(\mathtt{x}) \,\&\, \forall \mathtt{x}\ \mathrm{Q}(\mathtt{x})$
19. $\forall \mathtt{x}\ \mathrm{P}(\mathtt{x}) \,\&\, \forall \mathtt{x}\ \mathrm{Q}(\mathtt{x}) \to \forall \mathtt{x}\ (\mathrm{P}(\mathtt{x}) \,\&\, \mathrm{Q}(\mathtt{x}))$
20. $\exists \mathtt{x}\ (\mathrm{P}(\mathtt{x}) \vee \mathrm{Q}(\mathtt{x})) \to \exists \mathtt{x}\ \mathrm{P}(\mathtt{x}) \vee \exists \mathtt{x}\ \mathrm{Q}(\mathtt{x})$
21. $\exists \mathtt{x}\ \mathrm{P}(\mathtt{x}) \vee \exists \mathtt{x}\ \mathrm{Q}(\mathtt{x}) \to \exists \mathtt{x}\ (\mathrm{P}(\mathtt{x}) \vee \mathrm{Q}(\mathtt{x}))$
22. $\exists \mathtt{x}\ \mathrm{P}(\mathtt{x}) \to \neg\forall \mathtt{x}\ \neg\mathrm{P}(\mathtt{x})$
23. $\forall \mathtt{x}\ \mathrm{P}(\mathtt{x}) \to \neg\exists \mathtt{x}\ \neg\mathrm{P}(\mathtt{x})$
24. $\forall \mathtt{x}\ \exists \mathtt{y}\ \forall \mathtt{z}\ (\mathrm{P}(\mathtt{x}, \mathtt{y}) \to \mathrm{P}(\mathtt{y}, \mathtt{z}))$

Арифметика Пеано PA

Сигнатура: $\langle\{\mathbf{0}\}, \{+^{(2)}, \times^{(2)}, \mathbf{s}^{(1)}\}, \{=^{(2)}\}\rangle$

Аксиомы:

$A_{\times\mathbf{0}}:$ $\forall \mathtt{x}\ (\mathtt{x} \times \mathbf{0} = \mathbf{0})$

$A_{\times\mathbf{s}}:$ $\forall \mathtt{x}\ \forall \mathtt{y}\ (\mathtt{x} \times \mathbf{s}(\mathtt{y}) = \mathtt{x} \times \mathtt{y} + \mathtt{x})$

$A_{+\mathbf{0}}:$ $\forall \mathtt{x}\ (\mathtt{x} + \mathbf{0} = \mathtt{x})$

$A_{+\mathbf{s}}:$ $\forall \mathtt{x}\ \forall \mathtt{y}\ (\mathtt{x} + \mathbf{s}(\mathtt{y}) = \mathbf{s}(\mathtt{x} + \mathtt{y}))$

$A_{r=}:$ $\forall \mathtt{x}\ (\mathtt{x} = \mathtt{x})$

$A_{s=}:$ $\forall \mathtt{x}\ \forall \mathtt{y}\ ((\mathtt{x} = \mathtt{y}) \to (\mathtt{y} = \mathtt{x}))$

$A_{t=}:$ $\forall \mathtt{x}\ \forall \mathtt{y}\ \forall \mathtt{z}\ ((\mathtt{x} = \mathtt{y})\,\&(\mathtt{y} = \mathtt{z}) \to (\mathtt{x} = \mathtt{z}))$

$A_{=+\mathbf{s}}:$ $\forall \mathtt{x}\ \forall \mathtt{y}\ ((\mathtt{x} = \mathtt{y}) \to (\mathbf{s}(\mathtt{x}) = \mathbf{s}(\mathtt{y})))$

$A_{=-\mathbf{s}}:$ $\forall \mathtt{x}\ \forall \mathtt{y}\ ((\mathbf{s}(\mathtt{x}) = \mathbf{s}(\mathtt{y})) \to (\mathtt{x} = \mathtt{y}))$

$A_0:$ $\forall \mathtt{x}\ \neg(\mathbf{0} = \mathbf{s}(\mathtt{x}))$

$A_{ind}:$ $\varphi\,\{\mathtt{x}/\mathbf{0}\}\,\&\,\forall \mathtt{x}\ (\varphi \to \varphi\,\{\mathtt{x}/\mathbf{s}(\mathtt{x})\}) \to \forall \mathtt{x}\ \varphi$

$\mathbf{n}$ — сокращение для $\underbrace{\mathbf{s}(\mathbf{s}(\ldots\mathbf{s}}_{\mathbf{n}\text{ раз}}(\mathbf{0})\ldots))$, $n \in \mathbb{N}$

Упражнение 2

Предложить натуральный вывод формулы φ из подмножества множества PA или вывод секвенции $PA \vdash \varphi$:

1. $\mathbf{2} + \mathbf{0} = \mathbf{2}$
2. $\mathbf{2} + \mathbf{1} = \mathbf{3}$
3. $\mathbf{2} + \mathbf{2} = \mathbf{4}$
4. $\mathbf{2} \times \mathbf{0} = \mathbf{0}$
5. $\mathbf{2} \times \mathbf{1} = \mathbf{2}$
6. $\mathbf{2} \times \mathbf{2} = \mathbf{4}$
7. $\exists \mathtt{y}\ (\mathtt{x} = \mathbf{2} \times \mathtt{y}) \to \exists \mathtt{y}\ (\mathbf{s}(\mathtt{x}) = \mathbf{2} \times \mathtt{y} + \mathbf{1})$
8. $\forall \mathtt{x}\ (\mathtt{x} = 0 \vee \exists \mathtt{y}\ (\mathtt{x} = \mathbf{s}(\mathtt{y})))$